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Role of the interparticle interactions and axial rotation in the massive white dwarfs theory

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Using the equation of state of electron-nuclear model at high densities and the mechanical equilibrium equation we have investigated the influence of interparticle interactions and axial rotation on the macroscopic characteristics of massive white dwarfs. The method of solving the equilibrium equation in the case of rotation, using the basis of universal functions of the radial variable has been proposed. The conditions in which the axial rotation can compensate for weight loss of mass due to the interparticle Coulomb interactions have been established.

Key words: stars: white dwarfs, rotation, equation of state

INTRODUCTION

The discovery of white dwarfs at the beginning of the XX century [1] gave rise to the problem of the existence and stability of stars, which do not have sources of energy. R. Fowler's idea [7] that the existence of these objects is due to the pressure of the degenerate electron gas at high densities of matter led to formation of an electron-nuclear model. According to the latter, star consists of an ideal degenerate relativistic electron subsystem in the paramagnetic state at $T = 0\,\mathrm{K}$ and static nuclear subsystem, which is considered as a continuous classical environment [3, 5]. In the framework of this model the theory of cold white dwarfs was constructed by S. Chandrasekhar, the main consequences of which are restrictions on the mass ($M \leq 1.45 M_{\odot}$) and the peculiar "mass-radius" relation.

Due to intense observations performed in last century, it became known that white dwarfs have the same diversity of characteristics as the stars of other luminosity classes. The degeneration of electron subsystem is the main factor that determines the dwarf's electron structure and unites them. However, the Fowler-Chandrasekhar model is too idealized and can not explain all the diversity of the observed characteristics. The simplest example is the distribution of observed dwarfs on the "mass-radius" plane (see [19]). The important additional factors in formation of the structure of real white dwarfs are deviation from complete degeneration of electrons (finite temperature effects), Coulomb interparticle interactions, axial rotation, magnetic fields, effects of the general theory relativity and others. The generalizations of S. Chandrasekhar's model, that took into account some of these factors were proposed in the following works [9, 11, 14–16, 21].

In particular, R. James in his work [9] was the first, who showed that the influence of the axial rotation leads to increasing mass and ellipsoidal form of dwarfs with small and intermediate masses. The equation of state for electron-nuclear model of matter taking into account the interparticle interactions was first obtained by E. Salpeter [14]. In this case, the correlation energy of electron liquid model in the non-relativistic approximation and the energy of electron-nuclear interactions have been calculated in the Thomas-Fermi approximation. In this work it was established that the Coulomb interparticle interactions lead to a decrease of pressure of ideal relativistic electron gas. Due to the existence of dwarfs with large masses it is vital to perform more precise calculation of massive white dwarfs' structure considering the factors mentioned above. This task is related to the problem of stability of white dwarfs and the hypothesis that massive white dwarfs are precursors of type Ia supernovae.

According to the white dwarfs theory, magnetic field can influence their characteristics [2, 12]. In our work we consider non-magnetic massive white dwarfs, for which an adequate model can be easily constructed. As it follows from the observational data (see catalogue of white dwarfs from SDSS (WDMS) [13]), there are dwarfs with masses close to the S. Chandrasekhar's limit. The purpose of our work is simultaneous consideration the axial rotation and interparticle Coulomb interactions, which are competing factors in the theory of massive non-magnetic white dwarfs.

The first section describes a model of non-

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magnetic massive white dwarfs, within which the calculations of characteristics of dwarfs are performed. In the second section we calculate the equation of state in a homogeneous model that consists of fully degenerate relativistic electron gas at $T = 0\,\mathrm{K}$ in the presence of the static crystal lattice of nuclei within the multi-electron quantum-statistical approach. The obtained equation of state in the local approximation is used to write the equilibrium equation of star in the third section. Influence of the Coulomb interparticle interactions on the characteristics of a star without rotation are considered in section fourth. The simultaneous consideration of the interparticle interactions and rotation are investigated in the section five. Discussion of the calculation results and comparison with the observed data are given in the last section.

THE MODEL OF MASSIVE NON-MAGNETIC WHITE DWARF

It is well known that the S. Chandrasekhar's model is two-parametric, here appears the relativistic parameter in stellar center x_0 and the chemical composition parameter μ_e. By definition $x_0 = \hbar k_F(0)(m_0c)^{-1} = \hbar(3\pi^2 n(0))^{1/3}(m_0c)^{-1}$, where $n(0)$ is the number density of electrons in stellar center, m_0 is the electron mass, c is the speed of light. Because for dwarfs m_0c is the momentum scale, then x_0 really reflects the relativistic degree. The parameter x_0 is used in the theory of white dwarfs, where $x_0 \gtrsim 1$. From the definition follows that

$$x_0 = \frac{\eta}{r_s}\alpha_0,$$

where α_0 is the fine structure constant, $\eta = (9\pi/4)^{1/3}$, r_s is the Wigner-Brueckner parameter

$$r_s = \frac{r_0}{a_0} \equiv \frac{1}{a_0}\left(\frac{4\pi n(0)}{3}\right)^{-1/3},$$

where a_0 is the Bohr radius. In conventional metals, where is the subsystem of electrons conduction are degenerate, but non-relativistic, the parameter x_0 has an order of α_0 for $1.6 \leq r_s \leq 5.5$. Therefore, in metals, the electron density is represented by the parameter r_s, and in the white dwarfs – by the parameter x_0, which was introduced by S. Chandrasekhar. The chemical composition parameter $\mu_e = \langle A/z\rangle \approx 2.0$, ($A$ is the mass number of the nucleus, z is the nuclear charge) is the fraction of nuclear mass per one electron and formed by the full ionization of atom, which expressed in the atomic units of mass m_u.

We consider a model with four independent parameters x_0, μ_e, z and ω, where ω is the angular rotation velocity. Here μ_e, z and ω are considered as constants (z has the meaning of averaged charge in the presence of nuclei of various chemical elements). In the framework of this model, we have calculated the mass of dwarfs, which are close to the mass of observed massive dwarfs from WDMS binary catalogue [13] (see Table 1). For convenience we used here the scale of masses and radii:

$$M_0 = \left(\frac{3}{2}\right)^{1/2}\frac{1}{4\pi}\left(\frac{hc}{Gm_u^2}\right)^{3/2} m_u = \\ = 5.740247\cdot 10^{33} g \approx 2.88695 M_{\odot}, \quad (1)$$

$$R_0 = \left(\frac{3}{2}\right)^{1/2}\frac{1}{4\pi}\left(\frac{h^3}{cG}\right)^{1/2}\frac{1}{m_0 m_u} = \\ = 0.776885\cdot 10^{9} cm \approx 1.11623\cdot 10^{-2} R_{\odot}, \quad (2)$$

where m_0 is the electron mass, m_u is the atomic mass unit, c is the speed of light, G is the gravitational constant. The values of x_0 in Table 1 were found by solving the inverse problem – finding the parameters of S. Chandrasekhar's model using the observed radii and masses.

Table 1: The characteristics of massive white dwarfs from WDMS binary catalogue [13] and the relativistic parameter x_0 in the standard S. Chandrasekhar's model.

Number	R/R_0	$4M/M_0$	T_{eff}	x_0
64	0.44934	1.74596	16340	5.0087
1378	0.44934	1.75984	30071	5.0323
739	0.43234	1.76676	13857	5.2869
1391	0.42159	1.78756	16525	5.4904
1581	0.40548	1.8014	10189	5.7829
219	0.40101	1.81112	14472	5.8810
190	0.39385	1.81528	11433	6.0200
957	0.33924	1.88456	10793	7.3809
297	0.31776	1.9192	16836	8.0756
2195	0.30702	1.93996	11173	8.4718
33	0.27927	1.96768	13745	9.5720

THE EQUATION OF STATE OF HOMOGENEOUS ELECTRON-NUCLEAR MODEL

As usual, we consider a homogeneous model consisting of N_e electrons and $N_n = N_e/z$ nuclei with charge z in the volume V in the thermodynamic limit N_e, $N_n \to \infty$, $V \to \infty$, $N_e/V = const$ at absolute zero temperature, because the temperature effects are insignificant for massive white dwarfs. Static nuclei form a crystal lattice. This model corresponds to the Hamiltonian

$$\hat{H} = \hat{H}_0 + \frac{1}{2V}\sum_{\mathbf{q}\neq 0} V_{\mathbf{q}} \sum_{\mathbf{k}_1,\mathbf{k}_1}\sum_{s_1,s_1} a^{+}_{\mathbf{k}_1+\mathbf{q},s_1} a^{+}_{\mathbf{k}_2-\mathbf{q},s_2}\cdot$$
$$\cdot a_{\mathbf{k}_2,s_2} a_{\mathbf{k}_1,s_1} - \frac{z}{V}\sum_{\mathbf{q}\neq 0} V_{\mathbf{q}} S_{-\mathbf{q}} \sum_{\mathbf{k},s} a^{+}_{\mathbf{k}+\mathbf{q},s} a_{\mathbf{k},s} +$$
$$+ \frac{z^2}{2V}\sum_{\mathbf{q}\neq 0} V_{\mathbf{q}}\{S_{\mathbf{q}} S_{-\mathbf{q}} - N_n\}, \quad (3)$$

where

$$\hat{H}_0 = \sum_{\mathbf{k},s} E_k a^{+}_{\mathbf{k},s} a_{\mathbf{k},s}$$

is the Hamiltonian of an ideal model of relativistic electrons, $E_k = \{(m_0c^2)^2 + (\hbar kc)^2\}^{1/2} - m_0c^2$, $a^{+}_{\mathbf{k},s}(a_{\mathbf{k},s})$ are the creation and annihilation operators of electrons on the basis of plane waves ($\mathbf{k},\mathbf{q}$ are the wave vectors, s are the spin variables), $V_{\mathbf{q}} = 4\pi e^2/\mathbf{q}^2$ is the Fourier representation of Coulomb potential, $S_{\mathbf{q}} = \sum_{i=1}^{N_n} \exp[i(\mathbf{q},\mathbf{R}_j)]$ is structure factor of nuclear subsystem, $\mathbf{R}_j$ is the radius-vector of j-th nucleus. The considering model is electroneutral, therefore in the sum $\mathbf{q}$ there are no components with $\mathbf{q} = 0$. For the calculation of model energy (Eq. (3)) we have used the perturbation theory by powers of the operator of electron-nuclear interactions with the electron fluid model (the first two terms in Eq. (3)) as a zero approximation (basis model). The model energy (Eq. (3)) has the following representation [17]:

$$E(x|z) = E_e(x) + \frac{z^2}{2V}\sum_{\mathbf{q}\neq 0} V_{\mathbf{q}} S_{\mathbf{q}} S_{-\mathbf{q}} - N_n -$$
$$-\sum_{n\geq 2}\frac{z^n}{n!V^n}\sum_{\mathbf{q}_1,\ldots,\mathbf{q}_n\neq 0} V_{\mathbf{q}_1}V_{\mathbf{q}_2}\ldots V_{\mathbf{q}_n}\delta_{\mathbf{q}_1+\ldots+\mathbf{q}_n,0}\cdot$$
$$\cdot \mu_n(\mathbf{q}_1,\ldots,\mathbf{q}_n) S_{\mathbf{q}_1}\ldots S_{\mathbf{q}_n}, \quad (4)$$

where $E_e(x)$ is the energy of ground state of the basis model and $\mu_n(\mathbf{q}_1,\ldots,\mathbf{q}_n)$ are the static correlation n-particles functions of this model (statistic averages of the products of electron density operators). They are expressed by analogous correlation functions of the model of an ideal electron gas $\mu_n^{(0)}(\mathbf{q}_1,\ldots,\mathbf{q}_n)$ and the local field correction function (see [17, 18]). Since $\mu_n(\mathbf{q}_1,\ldots,\mathbf{q}_n) \sim N_e x^{n-1}(\hbar^2 k_F^2/2m_0)^{1-n}$ in the relativistic region, the n-th term in Eq. (4) has the order $N_e m_0 c^2 \alpha_0^n x$, where α_0 is the fine structure constant. This allows to consider only first terms of the series.

The energy of the basis model has a traditional representation [17]:

$$E_e(x) = N_e m_0 c^2\{\varepsilon_0(x) + \alpha_0\varepsilon_{HF}(x) + \alpha_0^2\varepsilon_c(x)\}, \quad (5)$$

where

$$\varepsilon_0(x) = (1+x^2)^{1/2} - 1 - (2x)^{-3}\mathcal{F}(x), \quad (6)$$

$$\mathcal{F}(x) = x(2x^2-3)(1+x^2)^{1/2} + $$
$$+ 3\ln[x + (1+x^2)^{1/2}] \quad (7)$$

is the contribution of the kinetic energy of electrons per particle in units m_0c^2, $\alpha_0\varepsilon_{HF}$ is the contribution of inter-electron interactions in the first order of perturbation theory ($\varepsilon_{HF}(x) = -3x/4\pi$) and $\alpha_0^2\varepsilon_c(x)$ is the so-called correlation energy in the same normalization. It is calculated numerically and represented by the following approximation [17]:

$$\varepsilon_c(x) = -\frac{b_0}{2}\int_0^x \frac{b_1 a + t^{1/2}}{t^{3/2} + t b_1 a + t^{1/2} b_2 a^2 + b_3 a^3}\times$$
$$\times \frac{1 + a_1 t + a_2 t^2}{1 + d_0 t}\mathrm{d}t, \quad (8)$$

where $a = (\alpha_0\eta)^{1/2}$, $\eta = (9\pi/4)^{1/3}$, $a_1 = 2.25328$, $a_2 = 4.87991$, $d_0 = 0.924022$, $b_0 = 0.0621814$, $b_1 = 9.81379$, $b_2 = 2.82214$, $b_3 = 0.69699$. All terms in Eq. (5) have linear asymptotic behavior with respect to the parameter x in the relativistic region ($x \geq 1$).

Electrostatic energy of static point nuclei, which compose a crystal lattice when a negative compensating background is present (the second term in Eq. (4)), has been calculated using the Ewald-Fuchs method [6, 8] as follows:

$$E_L(x|z) = -\frac{N_n\alpha}{2r_0}z^2e^2 = -N_e m_0 c^2\frac{\alpha}{2\eta}\alpha_0 x z^{2/3}, \quad (9)$$

where $r_0 = (3V/4\pi N_n)^{1/3}$, and α is the Madelung constant for a lattice of this type. We performed calculations for the volumetric cubic lattice ($\alpha = 1.79186$).

Since $S_{\mathbf{q}} = N_n\sum_{l\neq 0}\delta_{\mathbf{q},\mathbf{K}_l}$, the calculation of the so-called energy of the band structure (third term in Eq. (4)) reduces to the calculation of sums for nonzero vectors of the inverse lattice $\mathbf{K}_l$ (see [10]). In the approximation of two-electron correlations

$$-\frac{z^2}{2!V^2}\sum_{\mathbf{q}\neq 0} V^2_{\mathbf{q}}\mu_2(\mathbf{q},-\mathbf{q})S_{\mathbf{q}}S_{-\mathbf{q}} =$$
$$= N_e m_0 c^2\alpha_0^2 z^{4/3}\varepsilon_2(x|z),$$

and the results of numerical calculations for the volumetric cubic lattice can be approximated by expression

$$\varepsilon_2(x|z) = z^{1/6}\varepsilon_2(x),$$

$$\varepsilon_2(x) = -\{c_0 + c_1 x + c_2 x^2\}\{1 + d_1 x\}^{-1},$$

where $c_0 = 0.10582$, $c_1 = 0.11136$, $c_2 = 0.15535$, $d_1 = 1.29493$.

As can be seen in Eq. (5-9), all contributions to energy due to the Coulomb interactions are negative, which also leads to the ideal electron gas pressure decrease, because

$$P(x|z) = -\frac{\mathrm{d}E(x|z)}{\mathrm{d}V} = \frac{x^4}{N_e}\left(\frac{m_0 c}{\hbar}\right)^3 \frac{1}{9\pi^2}\frac{\mathrm{d}E(x|z)}{\mathrm{d}x} = \frac{\pi m_0^4 c^5}{3h^3}\left\{\mathcal{F}(x) - f(x|z)\right\}, \quad (10)$$

$$f(x|z) = \alpha_0 x^4 \left\{\frac{2}{\pi} + \frac{4\alpha}{3\eta} z^{2/3} - \frac{8}{3}\alpha_0\left[\frac{\mathrm{d}\varepsilon_c(x)}{\mathrm{d}x} + z^{4/3}\frac{\mathrm{d}\varepsilon_2(x|z)}{\mathrm{d}x}\right]\right\}. \quad (11)$$

Here $\mathrm{d}\varepsilon_c(x)/\mathrm{d}x, \mathrm{d}\varepsilon_2(x)/\mathrm{d}x < 0$ and have the same order of magnitude. Despite the differences in approach and other approximations (correlation energy in the non-relativistic approximation, the Thomas-Fermi equation for accounting electron-nuclear interactions, etc.) with E. Salpeter, our calculated expression for pressure is close to one found in the work [14].

THE GENERAL RELATIONS

The internal structure of a star with axial rotation is determined by the equilibrium equation [20]

$$\nabla P(\mathbf{r}) = -\rho(\mathbf{r})\nabla\left\{\Phi_{grav}(\mathbf{r}) + \Phi_c(\mathbf{r})\right\},$$

where

$$\Phi_{grav}(\mathbf{r}) = -G\int\frac{\rho(\mathbf{r}')\mathrm{d}\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|},$$

$$\Phi_c(\mathbf{r}) = -\frac{1}{2}\omega^2 r^2 \sin^2\theta$$

are the gravity and centrifugal potentials, $P(\mathbf{r})$ is the local pressure, θ and ω are the polar angle and constant angular velocity, respectively. The density of matter $\rho(\mathbf{r})$ is expressed in terms of the local value of the relativistic parameter $x(\mathbf{r})$,

$$\rho(\mathbf{r}) = m_u\mu_e n(\mathbf{r}) = \frac{m_u\mu_e}{3\pi^2}x^3(\mathbf{r})\left(\frac{m_0 c}{\hbar}\right)^3,$$

$$x(\mathbf{r}) = \frac{\hbar}{m_0 c}(3\pi^2 n(\mathbf{r}))^{1/3}.$$

where $n(\mathbf{r})$ is the number density of the electrons in the point $\mathbf{r}$.

To describe the star we used Eq. (10) and (2) in the local approximation by replacing x with its local value $x(\mathbf{r})$. In the dimensionless form $\xi = r/\lambda$, $Y(\xi,\theta) = \varepsilon_0^{-1}\{[1 + x^2(r)]^{1/2} - 1\}$, $\varepsilon_0 \equiv \varepsilon_0(x_0) = [1 + x_0^2]^{1/2} - 1$, $x_0 \equiv x(0)$, the equilibrium equation reduces to the differential equation

$$\Delta(\xi,\theta)Y(\xi,\theta) = \Omega^2 - \Gamma^3(\xi,\theta) + \varphi_1(\xi,\theta|z)\Delta(\xi,\theta)\Gamma(\xi,\theta) + \varphi_2(\xi,\theta|z)\times \times\left\{\left[\frac{\partial}{\partial\xi}\Gamma(\xi,\theta)\right]^2 + \frac{1-t^2}{\xi^2}\left[\frac{\partial}{\partial t}\Gamma(\xi,\theta)\right]^2\right\}. \quad (12)$$

To simplify the expression the following notations were used:

$$\Gamma(\xi,\theta) = \left[Y^2(\xi,\theta) + \frac{2}{\varepsilon_0}Y(\xi,\theta)\right]^{1/2}, \quad (13)$$

$$\varphi_1(\xi,\theta|z) = (2x)^{-3}\frac{\mathrm{d}f(x|z)}{\mathrm{d}x}\bigg|_{x=x(\xi,\theta)}, \quad (14)$$

$$\varphi_2(\xi,\theta|z) = \frac{\varepsilon_0}{8}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x^3}\frac{\mathrm{d}f(x|z)}{\mathrm{d}x}\right\}\bigg|_{x=x(\xi,\theta)}, \quad (15)$$

$$\Omega^2 = 2\frac{\omega^2 m_u\mu_e\lambda^2}{m_0 c^2\varepsilon_0}, \quad (16)$$

$$x(\xi,\theta) = \varepsilon_0\Gamma(\xi,\theta), \quad (17)$$

$$\Delta(\xi,\theta) = \Delta_\xi + \frac{1}{\xi^2}\Delta_\theta, \quad (18)$$

where

$$\Delta_\xi = \frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial}{\partial\xi}\right), \ \Delta_\theta = \frac{\partial}{\partial t}\left(1 - t^2\right)\frac{\partial}{\partial t}, \ t = \cos\theta.$$

In Eq. (13-18) parameter Ω is the dimensionless angular velocity of axial rotation, Δ_ξ is the radial component of Laplace operator in the dimensionless variables ξ and Δ_θ is the angular component of Laplace operator. The scale λ is determined by the expression

$$\frac{32\pi^2 G}{3(hc)^3}(m_u\mu_e m_0 c^2\lambda\varepsilon_0)^2 = 1.$$

Eq. (12) contains two independent parameters x_0, Ω, z and, in partial derivatives, satisfies the boundary conditions $Y(0,\theta) = 1$, $\partial Y(\xi,\theta)/\partial\xi = 0$ at $\xi = 0$. In order to highlight the analytical dependence on the relativistic parameter in stellar center we have introduced an approximate solution of Eq (12). Let us consider Eq. (12) without rotation, setting $\Omega = 0$ and replacing $Y(\xi,\theta)$ with the function $y(\xi)$ according to the spherical symmetry of the problem.

INFLUENCE OF THE INTERPARTICLE INTERACTIONS

The function $y(\xi)$ satisfies the one-dimensional differential equation

$$\Delta_\xi y(\xi) = -[\gamma(\xi)]^3 + \varphi_1(\xi|z)\Delta_\xi \gamma(\xi) + \varphi_2(\xi|z)\left\{\frac{\mathrm{d}}{\mathrm{d}\xi}\gamma(\xi)\right\}^2, \quad (19)$$

where $\gamma(\xi) = \left\{y^2(\xi) + 2y(\xi)\varepsilon_0^{-1}\right\}^{1/2}$. In Eq. (19), x_0 and z are the parameters and their functions $\varphi_i(\xi|z)$ are determined by Eq. (13-18), where one should make a replacement $x \to \varepsilon_0\gamma(\xi)$. Regular solutions of the equation satisfy the conditions $y(0) = 1$, $y'(0) = 0$. The condition $y(\xi) = 0$ determines the dimensionless radius of a star $\xi_1(x_0|z)$. Setting $\varphi_1 = \varphi_2 = 0$ the equation becomes one-parametrical equilibrium equation of S. Chandrasekhar's model (a standard model). The solutions of Eq. (19) were found numerically. The dependence $\xi_1(x_0|z)$ on the parameters x_0 and z is shown in Table 2, where dimensionless radius of a star in the standard model $\xi_1(x_0)$ is given for comparison. As can be seen from Table 2, accounting for interactions leads to the decrease of radius $\{\xi_1(x_0) - \xi_1(x_0|z)\}(\xi_1(x_0))^{-1}$ by 0.7% at $z = 2$, 1.2% at $z = 6$, 1.85% at $z = 12$ and 3% at $z = 26$.

Table 2: Dependence of the dimensionless radius $\xi_1(x_0|z)$ on the parameters x_0 and z ($\xi_1(x_0)$ corresponds to the standard model).

x_0	$\xi_1(x_0)$	$\xi_1(x_0\|z)$			
		$z=2$	$z=6$	$z=12$	$z=26$
1.0	1.03401	1.00101	0.98801	0.97401	0.94801
3.0	2.78201	2.74601	2.72401	2.70001	2.65601
5.0	3.70701	3.67001	3.64501	3.61701	3.56601
10.0	4.82801	4.79101	4.76401	4.73301	4.67601
15.0	5.35801	5.32201	5.29401	5.26301	5.20301
20.0	5.67001	5.63501	5.60701	5.57501	5.51501
30.0	6.02401	5.98901	5.96101	5.92901	5.86901

In this model the mass and radius of a star are determined by the expressions:

$$M(x_0|\mu_e|z) = \frac{M_0}{\mu_e^2}\mathcal{M}(x_0|z),$$

$$R(x_0|\mu_e|z) = \frac{R_0}{\mu_e}\frac{\xi_1(x_0|z)}{\varepsilon_0},$$

$$\mathcal{M}(x_0|z) = \int\limits_0^{\xi_1(x_0|z)} \xi^2\left\{y^2(\xi) + \frac{2}{\varepsilon_0}y(\xi)\right\}^{3/2}\mathrm{d}\xi,$$

where the mass and radius scales are determined by Eq. (1) and (2).

The dependence $\mathcal{M}(x_0|z)$ on the parameters x_0 and z and the dimensionless mass $\mathcal{M}(x_0)$ in the standard model are given in Table 3. The relative decrease of the mass induced by the influence of interactions $\{\mathcal{M}(x_0) - \mathcal{M}(x_0|z)\}(\mathcal{M}(x_0))^{-1}$ is approximately 1.4% at $z = 2$, 2.7% at $z = 6$, 4.1% at $z = 12$, 7% at $z = 26$ in the region $x_0 \geq 10$ (see Table 3).

For intermediate and large values of the relativistic parameter the function $f(x|z)$ is approximately proportional to x^4, consequently the expression $x^{-3}\mathrm{d}f/\mathrm{d}x$ is close to the constant value and its derivative over x is very small. This gives the opportunity to get an approximate estimate of the dwarf's characteristics without solving Eq. (19) numerically. In the core of the massive dwarf $x(r)$ is very close to the value x_0, the expression $x^{-3}(r)\mathrm{d}f(x(r))/\mathrm{d}x(r)$ can be approximately replaced by the $\varphi_1(x_0|z) = \mathrm{d}f(x_0|z)/\mathrm{d}x_0$. We have made sure that such replacement does not lead to a significant loss of accuracy.

INFLUENCE OF THE INTERPARTICLE INTERACTIONS AND AXIAL ROTATION

In the case $\Omega \neq 0$ Eq. (12) can be substantially simplified by neglecting the multiplier $\varphi_2(\xi,\theta|z)$, and replacing the term proportional to $\varphi_1(\xi,\theta|z)$ by $\varphi_1(x_0|z)\Delta(\xi,\theta)Y(\xi,\theta)$. One can introduce the dimensionless radial coordinate $\tilde{\xi} = r/\tilde{\lambda}$, where $\tilde{\lambda}$ is determined by the equation

$$\frac{32\pi^2 G}{3(hc)^3}(m_u\mu_e m_0 c^2\varepsilon_0\tilde{\lambda})^2 = 1 - \varphi_1(x_0|z),$$

then Eq. (12) takes the form

$$\Delta(\tilde{\xi},\theta)\tilde{Y}(\tilde{\xi},\theta) = \tilde{\Omega}^2 - \left\{\tilde{Y}^2(\tilde{\xi},\theta) + \frac{2}{\varepsilon_0}\tilde{Y}(\tilde{\xi},\theta)\right\}^{3/2}, \quad (20)$$

$$\tilde{\Omega}^2 = \frac{2\omega^2 m_u\mu_e}{m_0c^2\varepsilon_0}\tilde{\lambda}^2 = (1 - \varphi_1(x_0|z))\Omega^2. \quad (21)$$

Formally, Eq. (20) and (21) coincides with the equilibrium equation of the white dwarf with axial rotation in the standard model, written in the dimensionless form. The solution of Eq. (12) is

$$Y(\xi,\theta) \approx Y(\tilde{\xi}k,\theta),$$

where $\tilde{Y}(\tilde{\xi},\theta)$ is the solution of Eq. (20) and (21), and $k = [1 - \varphi_1(x_0|z)]^{1/2}$. In the case of massive white dwarf Eq. (20) and (21) have two small parameters – $\tilde{\Omega}$ and $\varepsilon_0^{-1} = [(1 + x_0^2)^{1/2} - 1]^{-1}$. In the limit $\tilde{\Omega} \to 0$,

Table 3: Dependence of the dimensionless mass $\mathcal{M}(x_0|z)$ on the parameters x_0 and z ($\mathcal{M}(x_0)$ corresponds to the standard model).

x_0	$\mathcal{M}(x_0)$	$\mathcal{M}(x_0\|z)$			
		$z=2$	$z=6$	$z=12$	$z=26$
1.0	0.707066	0.689037	0.673304	0.65581	0.624491
3.0	1.51862	1.49465	1.47331	1.44912	1.4045
5.0	1.76395	1.73843	1.71573	1.68996	1.64222
10.0	1.93284	1.90633	1.88277	1.85599	1.80626
15.0	1.97619	1.94943	1.92567	1.89863	1.84839
20.0	1.99337	1.96651	1.94268	1.91554	1.86508
30.0	2.00665	1.97972	1.95583	1.92861	1.87795

$x_0 \to \infty$ Eq. (20) and (21) transform to the equation of the polytropic model with index $n = 3$. At $\tilde{\Omega} \neq 0$ and $x_0 \to \infty$ Eq. (20) and (21) describe the equilibrium in a polytropic model (with index $n = 3$) of a white dwarf, rotating with constant angular velocity ω. Therefore the solution of Eq. (20) and (21) in the case of massive dwarfs can be represented in the form

$$\tilde{Y}(\xi,\theta) = y(\xi|x_0) + \\ + \tilde{\Omega}^2\{\Psi_0(\xi|x_0) + P_2(\cos\theta)\Psi_2(\xi|x_0)\}, \quad (22)$$

where $y(\xi|x_0)$ is the solution of the equilibrium equation for the white dwarf in the standard model, $P_2(\cos\theta)$ is Legendre polynomial and the functions $\Psi_0(\xi|x_0)$ and $\Psi_2(\xi|x_0)$ are the expansions in powers of the small parameter ε_0^{-1}:

$$\Psi_0(\xi|x_0) = \psi_{0,0}(\xi) + \sum_{l\geq 1} \psi_{l,0}(\xi)\varepsilon_0^{-l}(x_0),$$

$$\Psi_2(\xi|x_0) = A\psi_{0,2}(\xi) + \sum_{l\geq 1} \psi_{l,2}(\xi)\varepsilon_0^{-l}(x_0).$$

The functions $\psi_{l,0}(\xi)$ and $\psi_{l,2}(\xi)$ at $l \geq 1$, which are the solutions of the system of linear inhomogeneous one-dimensional differential equations of the variable ξ, were found in this work by numerical integration. The functions $\psi_{0,0}(\xi)$ and $A\psi_{0,2}(\xi)$ were calculated in [4].

The conditions $\tilde{Y}(\xi,\pi/2) = 0$, $\partial\tilde{Y}(\xi,\pi/2)/\partial\xi = 0$ determine the maximal value of the angular velocity $\tilde{\Omega}_{max}(x_0)$ and corresponding maximal value of the dimensionless equatorial radius $\xi_e^{max}(x_0)$. At $\tilde{\Omega} > \tilde{\Omega}_{max}$ the stability of a star is disturbed in the vicinity of the equator and the function $\tilde{Y}(\xi,\theta)$ becomes a non-monotonous function of ξ. The root of the equation $\tilde{Y}(\xi,\theta) = 0$ at $\tilde{\Omega} < \tilde{\Omega}_{max}$ determines the shape of a star $\xi_1(\theta) \equiv \xi_1(\theta|x_0,\tilde{\Omega})$. The dependence $\tilde{\Omega}^2_{max}$ on the relativistic parameter x_0, as well as the equatorial $\xi_e(x_0|\tilde{\Omega}^2_{max})$ and polar $\xi_p(x_0|\tilde{\Omega}^2_{max})$ radii as the functions of x_0 are given in Table 4.

Table 4: Dependence of the maximal value of parameter $\tilde{\Omega}^2$ on the parameter x_0, corresponding dimensionless equatorial and polar radii, as well as the dimensionless radius of the dwarf without rotation $\xi_1(x_0)$.

x_0	$\tilde{\Omega}^2_{max}$	$\xi_1(x_0)$	$\xi_e(x_0\|\tilde{\Omega}^2_{max})$	$\xi_p(x_0\|\tilde{\Omega}^2_{max})$
6.0	0.0164	4.023	5.401	3.801
8.0	0.0119	4.493	6.001	4.271
10.0	0.00972	4.828	6.461	4.591
15.0	0.00733	5.358	7.231	5.111
20.0	0.00632	5.670	7.651	5.421
25.0	0.00579	5.887	8.021	5.631

The approximate solution of Eq. (12) has been obtained by replacing $\xi \to k\xi$ in Eq. (22), where $k = [1 - \varphi_1(x_0|z)]^{1/2}$. Therefore, the mass of white dwarf in a model with interactions is determined by the expressions

$$M(x_0,\mu_e,z,\omega) = \frac{M_0}{\mu_e^2}(1 - \varphi_1(x_0|z))^{3/2}\mathcal{M}(x_0|\tilde{\Omega}),$$

$$\mathcal{M}(x_0|\tilde{\Omega}) = \\ = \int\limits_0^1 \mathrm{d}t \int\limits_0^{\xi_1(\theta|x_0,\tilde{\Omega})} \xi^2\left\{\tilde{Y}^2(\xi,\theta) + \frac{2}{\varepsilon_0}\tilde{Y}(\xi,\theta)\right\}^{3/2}\mathrm{d}\xi.$$

The dependence on the x_0 of the maximal mass with rotation (but without the interactions) $\mathcal{M}(x_0,\tilde{\Omega}_{max})$ and the mass in the model with rotation and interactions $[1-\varphi_1(x_0,z)]^{3/2}\mathcal{M}(x_0,\tilde{\Omega}_{max})$ are given in Table 5.

Table 5: Dependence of the dimensionless dwarf's mass on the parameter x_0 in different approximations.

x_0	$\mathcal{M}(x_0,\tilde{\Omega}_{max})$	$(1-\varphi_1(x_0,z))^{3/2}\mathcal{M}(x_0,\tilde{\Omega}_{max})$		
		$z=2$	$z=12$	$z=26$
5.0	1.863482	1.838623	1.792012	1.747669
7.0	1.968121	1.941846	1.892602	1.845721
10.0	2.033656	2.006491	1.955589	1.907083
15.0	2.068962	2.041313	1.989509	1.940093
20.0	2.081174	2.053355	2.001235	1.951486
30.0	2.088186	2.060267	2.007960	1.957998

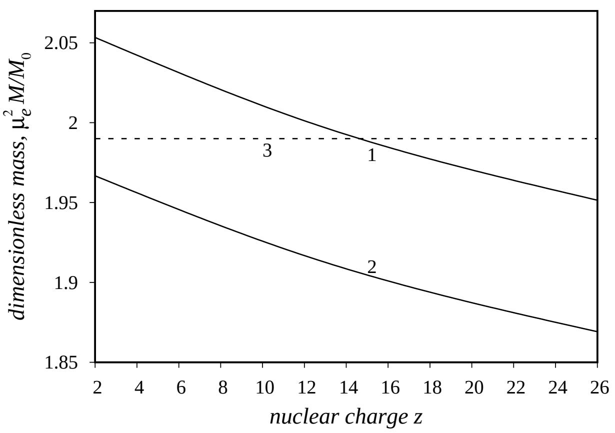

Fig. 1: Dependence of the dimensionless dwarf's mass on nuclear charge z at $x_0=20$ (curve 1 corresponds to the model with the interparticle interactions and maximal rotation velocity, curve 2 illustrates mass dependence of dwarf without rotation, curve 3 – result of S. Chandrasekhar's model).

CONCLUSIONS

The main factor of formation of non-magnetic white dwarfs is the degeneration of electron subsystem which limits their maximal mass (S. Chandrasekhar's limit $M_{max}=\mu_e^{-2}M_0\cdot 2.01824$). Among additional factors there are two the most important – the interparticle interactions and axial rotation. These factors are competing ones and change S. Chandrasekhar's limit. The latter becomes the function of chemical composition and maximal rotation velocity, which in term depends on x_0 and z. Rotation of dwarf leads to increase of its mass and gaining the ellipsoidal configuration. The deviation of polar and equatorial radii in S. Chandrasekhar's model increases with increasing relativistic parameter in stellar center x_0. The interparticle interactions cause the mass decrease and ellipsoidal shape of dwarf. In this approach, in contrast to S. Chandrasekhar's model, the influence of the chemical composition reveals itself. This can be seen in Fig. 1, were the dependence of dwarf's mass on maximal rotation velocity is shown for the case $x_0=20$. The axial rotation can partially (depending on the value $\tilde{\Omega}$) compensate the influence of the interparticle Coulomb interactions at $z\lesssim 15$. In this case the dwarf mass can exceed S. Chandrasekhar's limit. In the region $z>15$ this compensation is generally impossible, so the masses of such dwarfs can not exceed this limit. Our results are in good agreement with the results obtained from observations in Table 1. For example, as can be seen from Table 5, mass of a dwarf in the model with $x_0=10$ varies from 2.006491 for $z=2$ to 1.955589 for $z=12$, which agrees with mass of dwarf with number 33 from Table 1.

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